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THE STABILITY OF ORTHOTROPIC CYLINDRICAL SHELLS

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This paper presents a study of the stability of open circular cylindrical shells (panels) subjected to compression along the generator. It is assumed that the shell is orthotropic. This is the case with closely spaced longitudinal and lateral stiffeners if they are considered to be "spread" over the entire surface of the shell. The closer the stiffeners are spaced, the more accurate this assumption will be. Structures also make use of shells of materials having anisotropic elastic properties (for example, plywood, plastics, reinforced materials, sandwich skins, etc.), which may be considered to be orthotropic.

The article considers the stability of the panel "in the large": the buckling deflections are considered to be comparable to the thickness of the shell. This leads to the consideration of the nonlinear problem and the associated complex solution. However, this complexity is unavoidable; we know that the solutions of the problem of the stability of shells "in the small", based on linear theory, are not confirmed experimentally.

On the basis of the results presented in this paper, it is possible, using the known geometric parameters of the panel and the elastic properties of the material, to determine the critical compressive stresses for the panel and to determine its load-carrying capability.

It should be remarked that the results of the first four sections are based on assumptions concerning the pattern of the wave formation which must be checked experimentally. Section 5 presents a somewhat refined solution.

This study is an extension of the investigations of A. S. Vol'mir on the stability of isotropic cylindrical panels (Ref. 3).

1. FUNDAMENTAL RELATIONS OF THE THEORY OF THE FLEXIBLE CIRCULAR CYLINDRICAL ORTHOTROPIC SHELL

We will consider the portion of the circular cylindrical orthotropic shell bounded by two arcs and two generators.

Assume that the material of the shell at each point possesses three mutually perpendicular planes of elastic symmetry and that at each point, one of the planes of elastic symmetry is tangential to the middle surface.

In the general case, orthotropic shells with material properties variable across the shell thickness are considered. Such shells include shells of sandwich materials, panels stiffened by longitudinal and lateral ribs, etc. /500

We will introduce the following notation (Fig. 1):

r - radius of curvature of the shell;

θ - central angle subtended by the generating arc;

h - thickness of the shell;

a, b - dimensions of the shell along the generator and along the arc, respectively.

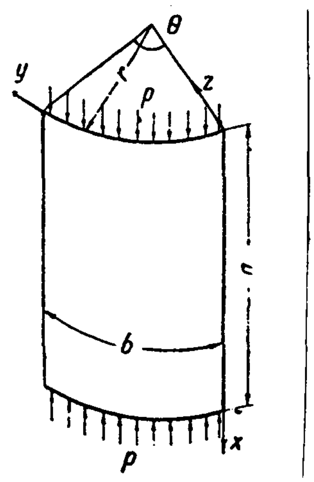

Fig. 1

The coordinate axes x, y, and z will be directed, respectively, along the generator, along the tangent to the generating arc, and toward the center of curvature. The displacements of the points of the middle surface along the coordinate axes will be denoted by u, v, and w.

We will consider the displacements u, v to be small in comparison with the thickness of the shell, while the deflections w may be comparable to the thickness.

The thickness of the shell is small in comparison with the radius of curvature r.

The deformed state of the shell is characterized by six quantities -- the three components of the deformation of the middle ϵ_x^0, ϵ_y^0, and γ_{xy}^0 and the three deviations of the curvature of the middle surface χ_x, χ_y, and χ_{xy} -- which, under the stated assumptions, have the form

$$\left.\begin{aligned} \epsilon_x^0 &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2; \quad \chi_x = -\frac{\partial^2 w}{\partial x^2}, \\ \epsilon_y^0 &= \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - \frac{w}{r}; \quad \chi_y = -\frac{\partial^2 w}{\partial y^2}, \\ \gamma_{xy}^0 &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}; \quad \chi_{xy} = -\frac{\partial^2 w}{\partial x \partial y}, \end{aligned}\right\} \tag{1}$$

i.e., the six components of the deformation are expressed in terms of the derivatives of the three functions, u, v, w along the x, y, z coordinates. Therefore, the components of the deformation are not independent functions of x, y, z but must be related by differential relations. Differentiating the expressions in the left-hand column of (1), we obtain one of these relations -- the equation of compatibility of the deformations

$$\frac{\partial^2 \epsilon_x^0}{\partial y^2} + \frac{\partial^2 \epsilon_y^0}{\partial x^2} - \frac{\partial^2 \gamma_{xy}^0}{\partial x \partial y} = \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2} - \frac{1}{r}\cdot\frac{\partial^2 w}{\partial x^2}. \tag{2}$$

For the orthotropic shell which is homogeneous across its width, the properties of elasticity are characterized by four independent parameters: E_1 and E_2, the longitudinal elasticity moduli along the generator and along the arc of the shell; G, the shear modulus; μ_1 the Poisson coefficient.

The cylindrical and the torsional stiffnesses of the shell are expressed in terms of these parameters as follows:

$$D_1 = \frac{E_1 h^3}{12(1 - \mu_1\mu_2)}\ ; \quad D_2 = \frac{E_2 h^3}{12(1 - \mu_1\mu_2)}\ ; \quad D_k = \frac{Gh^3}{12}\,. \tag{3}$$

The following known relation holds: /501

$$\mu_1 E_2 = \mu_2 E_1. \tag{4}$$

The elastic properties of an orthotropic shell which is not uniform across its thickness are characterized by the following values, which are averaged (reduced) across the thickness: the moduli of elasticity and the Poisson coefficient, E_1^0, E_2^0, G^0, and μ_1^0, which characterize tension-compression and shear; and the four quantities, E_1, E_2, G_0, and μ_1, which characterize the elasticity of the shell in bending and torsion. The subscript 1 corresponds to the x-axis and the subscript 2 to the y-axis.

We will express the components of the stressed state in terms of the components of the deformation. The total deformation at any point of the shell is composed of two types: the deformations in the middle surface ϵ_x^0, ϵ_y^0, γ_{xy}^0 and the bending deformations ϵ_x', ϵ_y', γ_{xy}'.

The stressed state of the shell is characterized by eight quantities: the three stresses in the middle surface σ_x^0, σ_y^0, and τ_{xy}^0; three moments; M_x, the bending moment per unit length of arc of the shell cross-section; M_y, the bending moment per unit length of the generator; H_{xy}, the torsional moment per unit length of the section; and the two shearing forces Q_x and Q_y.

Since we will limit ourselves to the proportional region, we will use Hooke's law to relate the deformations and the stresses (at the same time, we take $\sigma_z = 0$);

$$\left.\begin{aligned}\epsilon_x &= \frac{1}{E_x}(\sigma_x - \mu_x \sigma_y) ; \\ \epsilon_y &= \frac{1}{E_y}(\sigma_y - \mu_y \sigma_x) ; \\ \gamma_{xy} &= \frac{1}{G}\tau_{xy},\end{aligned}\right\} \tag{5}$$

where E_x (z), E_y (z) - elastic moduli along the principal axes, variable across the thickness;
G_z - shear modulus;
μ_x (z), μ_y (z) - Poisson coefficients for the principal axes.

For tension and shear in the middle surface of a nonuniform orthotropic shell, the relation between the mean stresses and the deformations has the form

$$\left.\begin{aligned}\epsilon_x^0 &= \frac{1}{E_1^0}(\sigma_x^0 - \mu_1^0 \sigma_y^0) ; \\ \epsilon_y^0 &= \frac{1}{E_2^0}(\sigma_y^0 - \mu_2^0 \sigma_x^0) ; \\ \gamma_{xy}^0 &= \frac{1}{G^0}\tau_{xy}^0,\end{aligned}\right\} \tag{6}$$

where the elastic parameters averaged across the thickness are determined from the condition of equality of the forces for the nonuniform shell with elastic constants E_x, E_y, G, μ_x, μ_y to those for the uniform shell of the same thickness with constants E_1^0, E_2^0, G^0, μ_1^0, μ_2^0 for identical deformations:

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$$\left.\begin{aligned}\int_{-h/2}^{h/2} \sigma_x dz &= h\sigma_x^0 ;\\ \int_{-h/2}^{h/2} \sigma_y dz &= h\sigma_y^0 ;\\ \int_{-h/2}^{h/2} \tau_{xy} dz &= h\tau_{xy}^0 .\end{aligned}\right\} \quad (7)$$

Whence, considering (5), (6), and the equality of the deformations, we obtain

$$\left.\begin{aligned}\frac{E_1^0}{1-\mu_1^0\mu_2^0} &= \frac{1}{h}\int_{-h/2}^{h/2} \frac{E_x}{1-\mu_x\mu_y}\,dz ;\\ \frac{E_2^0}{1-\mu_1^0\mu_2^0} &= \frac{1}{h}\int_{-h/2}^{h/2} \frac{E_y}{1-\mu_x\mu_y}\,dz ;\\ \mu_1^0 &= \frac{1-\mu_1^0\mu_2^0}{E_2^0 h}\int_{-h/2}^{h/2} \frac{E_y\mu_x}{1-\mu_x\mu_y}\,dz ;\\ \mu_2^0 &= \frac{1-\mu_1^0\mu_2^0}{E_1^0 h}\int_{-h/2}^{h/2} \frac{E_x\mu_y}{1-\mu_x\mu_y}\,dz ;\\ G^0 &= \frac{1}{h}\int_{-h/2}^{h/2} G dz ; \quad \mu_1^0 E_2^0 = \mu_2^0 E_1^0 .\end{aligned}\right\} \quad (8)$$

On the basis of the hypothesis of straight normals and (1), the components of the deformations in bending and torsion will be

$$\epsilon'_x = -z\frac{\partial^2 w}{\partial x^2};\quad \epsilon'_y = -z\frac{\partial^2 w}{\partial y^2};\quad \gamma'_{xy} = -2z\frac{\partial^2 w}{\partial x\partial y}. \tag{9}$$

From Hooke's law (5),

$$\left.\begin{aligned}
\sigma'_x &= \frac{E_x}{1-\mu_y\mu_y}(\epsilon'_x + \mu_y\epsilon'_y) = -\frac{E_x z}{1-\mu_x\mu_y}\left(\frac{\partial^2 w}{\partial x^2} + \mu_y\frac{\partial^2 w}{\partial y^2}\right);\\
\sigma'_y &= \frac{E_y}{1-\mu_x\mu_y}(\epsilon'_y + \mu_x\epsilon'_x) = -\frac{E_y z}{1-\mu_x\mu_y}\left(\frac{\partial^2 w}{\partial y^2} + \mu_x\frac{\partial^2 w}{\partial x^2}\right);\\
\tau'_{xy} &= G\gamma'_{xy} = -2Gz\frac{\partial^2 w}{\partial x\partial y}.
\end{aligned}\right\} \tag{10}$$

The bending and torsional moments per unit length, corresponding to the stresses σ'_x, σ'_y, τ'_{xy}, are

$$M_x = \int_{-h/2}^{h/2}\sigma'_x z\,dz;\quad M_y = \int_{-h/2}^{h/2}\sigma'_y z\,dz;\quad H_{xy} = \int_{-h/2}^{h/2}\tau'_{xy} z\,dz. \tag{11}$$

Substituting expression (10) here, we obtain /503

$$\left.\begin{aligned}
M_x &= -\int_{-h/2}^{h/2}\frac{E_x z^2}{1-\mu_x\mu_y}\left(\frac{\partial^2 w}{\partial x^2} + \mu_y\frac{\partial^2 w}{\partial y^2}\right)dz = -D_1\left(\frac{\partial^2 w}{\partial y^2} + \mu_2\frac{\partial^2 w}{\partial y^2}\right);\\
M_y &= -\int_{-h/2}^{h/2}\frac{E_y z^2}{1-\mu_x\mu_y}\left(\frac{\partial^2 w}{\partial y^2} + \mu_x\frac{\partial^2 w}{\partial x^2}\right)dz = -D_2\left(\frac{\partial^2 w}{\partial y^2} + \mu_1\frac{\partial^2 w}{\partial x^2}\right);\\
H_{xy} &= -\int_{-h/2}^{h/2}2Gz^2\frac{\partial^2 w}{\partial x\partial y}\,dz = -2D_k\frac{\partial^2 w}{\partial x\partial y},
\end{aligned}\right\} \tag{12}$$

where D_1, D_2, D_k - cylindrical stiffnesses of the orthotropic shell in bending and torsion;

μ_1, μ_2 - reduced coefficients of lateral deformation in bending.

$$\left.\begin{aligned} D_1 &= \int_{-h/2}^{h/2} \frac{E_x z^2}{1 - \mu_x\mu_y}\,dz; \quad D_2 = \int_{-h/2}^{h/2} \frac{E_y z^2}{1 - \mu_x\mu_y}\,dz; \quad D_k = \int_{-h/2}^{h/2} Gz^2 dz; \\ \mu_1 &= \frac{1}{D_2}\int_{-h/2}^{h/2} \frac{E_y\mu_x z^2}{1 - \mu_x\mu_y}\,dz; \quad \mu_2 = \frac{1}{D_1}\int_{-h/2}^{h/2} \frac{E_x\mu_y z^2}{1 - \mu_x\mu_y}\,dz. \end{aligned}\right\} \quad (13)$$

By analogy with the uniform orthotropic shell of the same thickness, the quantities D_1, D_2, D_k may also be expressed in the form

$$D_1 = \frac{E_1 h^3}{12(1 - \mu_1\mu_2)}; \quad D_2 = \frac{E_2 h^3}{12(1 - \mu_1\mu_2)}; \quad D_k = \frac{G_0 h^3}{12}. \quad (14)$$

from which, with consideration of (13), we may find the reduced values of the elastic parameters of the nonuniform shell in bending and torsion.*

From the equilibrium of an infinitesimal parallelepiped cut from the shell by planes perpendicular to the x- and y-axes, we can find the equation of equilibrium. First, we will introduce the stress function F, satisfying the equalities

$$\frac{\partial^2 F}{\partial y^2} = \sigma_x^0; \quad \frac{\partial^2 F}{\partial x^2} = \sigma_y^0; \quad \frac{\partial^2 F}{\partial x \partial y} = -\tau_{xy}^0. \quad (15)$$

The equilibrium equation has the form

$$D_1 \frac{\partial^4 w}{\partial x^4} + 2D_3 \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w}{\partial y^4} =$$

$$= h\left[\frac{\partial^2 w}{\partial y^2}\cdot\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 F}{\partial x^2}\left(\frac{\partial^2 w}{\partial y^2} + \frac{1}{r}\right) - 2\,\frac{\partial^2 F}{\partial x \partial y}\cdot\frac{\partial^2 w}{\partial x \partial y}\right]. \quad (16)$$

*Equations (8) and (13) were obtained by G. G. Rostovtsev (Ref. 2).

Here,

$$2D_3 = D_1\mu_2 + D_2\mu_1 + 4D_k. \tag{17}$$

The equation for the compatibility of the deformations (2), /504 with consideration of (6) and (15), takes the form

$$K_2 \frac{\partial^4 F}{\partial x^4} + 2K_3 \frac{\partial^4 F}{\partial x^2 \partial y^2} + K_1 \frac{\partial^4 F}{\partial y^4} = \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} - \frac{1}{r} \cdot \frac{\partial^2 w}{\partial x^2}, \tag{18}$$

where

$$K_2 = \frac{1}{E_2^0}; \quad K_1 = \frac{1}{E_1^0}; \quad 2K_3 = \frac{1}{G_0} - \frac{2\mu_1^0}{E_1^0}. \tag{19}$$

Equations (16) and (18) differ from the equations for the flexible plate obtained by Rostovtsev by the terms which contain the factor $1/r$, the curvature of the shell.

The exact solution of the system of equations (16) and (18) is scarcely possible. We will use the Ritz method for solution of the problem. To do this, we need an expression for the potential energy of deformation of the shell.

We write the expression for the specific potential energy (Ref. 1):

$$\overline{U} = \frac{1}{2}\left(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx} + \tau_{xy}\gamma_{xy}\right). \tag{20}$$

The potential energy U of the entire body of volume V

$$U = \iiint_V \overline{U}\, dx\, dy\, dz. \tag{21}$$

Since we are considering a very thin shell, the stress $\sigma_z = 0$ and the deformations $\gamma_{yz} = \gamma_{zx} = 0$, and consequently,

$$U = \frac{1}{2} \int\int\int_V (\sigma_x \epsilon_x + \sigma_y \epsilon_y + \tau_{xy}\gamma_{xy})\, dx\, dy\, dz. \tag{22}$$

The potential energy of deformation of the shell consists of the energy of bending and torsion deformation, U_{bend}, and the energy of elongation and shear, U_{sh}

$$U = U_{bend} + U_{sh}. \tag{23}$$

Here

$$U_{bend} = \frac{1}{2} \int\int\int_V (\sigma'_x\, \epsilon'_x + \sigma'_y\, \epsilon'_y + \tau'_{xy}\, \gamma'_{xy})\, dx\, dy\, dz \tag{24}$$

and

$$U_{sh} = \frac{1}{2} \int\int\int_V (\sigma^0_x\, \epsilon^0_x + \sigma^0_y\, \epsilon^0_y + \tau^0_{xy}\, \gamma^0_{xy})\, dx\, dy\, dz. \tag{25}$$

Substituting in (24) the bending stresses from equation (10) and the components of the deformation from equation (9), we have:

$$U_{bend} = \frac{1}{2} \int_S\int\int_{-h/2}^{h/2} \left\{ \frac{1}{1 - \mu_x\mu_y} \left[E_x \left(\frac{\partial^2 w}{\partial x^2}\right)^2 z^2 + E_y \left(\frac{\partial^2 w}{\partial y^2}\right)^2 z^2 + \right.\right.$$

$$\left.\left. + (E_x\mu_y + E_y\mu_x) \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} z^2 \right] + 4G \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 z^2 \right\} dz\, dx\, dy =$$

$$= \frac{1}{2} \int_S\int \left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \int_{-h/2}^{h/2} \frac{E_x z^2}{1 - \mu_x\mu_y} dz + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 \int_{-h/2}^{h/2} \frac{E_y z^2}{1 - \mu_x\mu_y} dz + \right.$$ /505

$$\left. + \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} \int_{-h/2}^{h/2} \frac{2E_y\mu_x z^2}{1 - \mu_x\mu_y} dz + 4 \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \int_{-h/2}^{h/2} G z^2 dz \right] dx\, dy. \tag{26}$$

Using expressions (13) for the reduced elastic parameters, we obtain

$$U_{bend} = \frac{1}{2}\int_S\int\left[D_1\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + D_2\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + \right.$$

$$\left. + 2\mu_1 D_2 \frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2} + 4D_k\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right] dx\ dy. \quad (27)$$

We will now substitute into (25) the values of the deformation from equations (6):

$$U_{sh} = \frac{1}{2}\int\int_V\int\left[\frac{1}{E_1^0}\left(\sigma_{x}^{0\,2} - \mu_1^0\,\sigma_x^0\,\sigma_y^0\right) + \right.$$

$$\left. + \frac{1}{E_2^0}\left(\sigma_{y}^{0\,2} - \mu_2^0\,\sigma_x^0\,\sigma_y^0\right) + \frac{\tau_{xy}^0}{G^0}\right] dz\ dx\ dy. \quad (28)$$

Here, we introduce the stress function for the middle surface from equations (15). After integration with respect to z, we obtain

$$U_{sh} = \frac{h}{2}\int_S\int\left[\frac{1}{E_2^0}\left(\frac{\partial^2 F}{\partial x^2}\right)^2 + \frac{1}{E_1^0}\left(\frac{\partial^2 F}{\partial y^2}\right)^2 - \right.$$

$$\left. - \frac{2\mu_1^0}{E_1^0}\cdot\frac{\partial^2 F}{\partial x^2}\cdot\frac{\partial^2 F}{\partial y^2} + \frac{1}{G^0}\left(\frac{\partial^2 F}{\partial x \partial y}\right)^2\right] dx\ dy. \quad (29)$$

2. LARGE DEFLECTIONS OF THE CIRCULAR CYLINDRICAL ORTHOTROPIC SHELL UNDER COMPRESSION ALONG THE GENERATOR

We will introduce the expression for the deflection w at shell buckling in the form of the series

$$w = f_1\eta_1 + f_2\eta_2 + f_3\eta_3 + \cdots \quad (30)$$

Here, $\eta_1, \eta_2, \eta_3, \ldots$ - certain functions of the x- and y-coordinates, each of which satisfies the geometric boundary conditions;

$f_1, f_2, f_3, \ldots$ - parameters to be defined.

We now introduce the expression for the energy of the system = U - W, where U is the potential energy of deformation and W is the work of the external forces as a function of the parameters f_1, f_2, f_3, ... Now, from equation (18), substituting the value of the deflection w in the form of the series (30), we obtain the stress function F as a function of the parameters f_1, f_2, f_3, ...

The equilibrium states of the system are determined from the condition of minimum system energy: /506

$$\frac{\partial(U - W)}{\partial f_1} = 0; \quad \frac{\partial(U - W)}{\partial f_2} = 0; \quad \frac{\partial(U - W)}{\partial f_3} = 0; \quad \cdots \qquad (31)$$

The system of equations obtained by the Ritz method contains the unknown parameters f_1, f_2, f_3, ··· The number of these parameters is equal to the number of equations in the system (31). Thus, the solution of the problem is reduced to the system of algebraic equations (31).

As an expression for the deflection at buckling, we will use, for the orthotropic shell, the same expression which was used by A. S. Vol'mir in the study of the large deflections of the isotropic shell (Ref. 3).

We will assume that the panel is long, i.e., that the dimension along the generator is greater than the width. We will consider that the edges of the panel are hinged on rigid members (it should be noted that the constraint conditions have very little effect on the critical stress of shells of significant curvature).

We will consider that the shell buckles at loss of stability are of equal dimensions in the directions of the generator and the arc, i.e., that they are nearly circular. This assumption simplifies the study of the system (31).

We will now consider the expression for the deflection which reflects the ellipticity of the buckle. This will be a refinement of the solution which will more accurately reflect the actual nature of the buckling.

We will limit ourselves to two terms of the series (30):

$$w = f_1 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} + f_2 \sin^2 \frac{m\pi x}{a} \sin^2 \frac{n\pi y}{b} . \qquad (32)$$

Since we will consider circular buckles, $b = a$, where a is the length of m halfwaves along the generator; $m = n$, since we are considering a long panel; n will be considered to be variable.

Thus, as an expression for the deflection, we will take

$$w = f_1 \sin \frac{n\pi x}{a} \sin \frac{n\pi y}{a} + f_2 \sin^2 \frac{n\pi x}{a} \sin^2 \frac{n\pi y}{a} \qquad (33)$$

or

$$w = f_1 \sin \alpha \sin \beta + f_2 \sin^2 \alpha \sin^2 \beta, \qquad (34)$$

where

$$\alpha = \frac{n\pi x}{a}; \quad \beta = \frac{n\pi y}{a} . \qquad (35)$$

Substituting (34) into (18), we obtain the expression for the stress function

$$F = \frac{1}{32K_2} (f_1^2 + f_2^2) \cos 2\alpha + \frac{1}{32K_1} (f_1^2 + f_2^2) \cos 2\beta -$$

$$- \frac{1}{512K_2} f_2^2 \cos 4\alpha - \frac{1}{512K_1} f_2^2 \cos 4\beta + \frac{1}{32x_2} f_2^2 \cos 2\alpha \cos 4\beta +$$

$$+ \frac{1}{32x_3} f_2^2 \cos 4\alpha \cos 2\beta - \frac{1}{16x_1} f_2^2 \cos 2\alpha \cos 2\beta +$$

$$+ \frac{3}{2x_4} f_1 f_2 \sin \alpha \sin 3\beta + \frac{3}{2x_5} f_1 f_2 \sin 3\alpha \sin \beta -$$

$$- \frac{1}{x_1} f_1 f_2 \sin \alpha \sin \beta - \frac{a^2}{16K_2 r\pi^2 n^2} f_2 \cos 2\alpha +$$ /508

$$+ \frac{a^2}{16x_1 r\pi^2 n^2} f_2 \cos 2\alpha \cos 2\beta + \frac{a^2}{r x_1 \pi^2 n^2} f_1 \sin \alpha \sin \beta - \frac{p_0 y^2}{2}, \qquad (36)$$

where p_0 is the external compressive stress applied to the ends of the shell;

$$\left.\begin{aligned} x_1 &= K_2 + 2K_3 + K_1; \\ x_2 &= K_2 + 8K_3 + 16K_1; \\ x_3 &= 16K_2 + 8K_3 + K_1; \\ x_4 &= K_2 + 18K_3 + 81K_1; \\ x_5 &= 81K_2 + 18K_3 + K_1. \end{aligned}\right] \tag{37}$$

We will determine the energy Э of the system for the segment of the panel of length 2a along the generator and width b along the arc:

$$Э = U - W = U_{bend} + U_{sh} - W. \tag{38}$$

Substituting in (27) the expression for the deflection (33) and integrating over the surface of the shell S, i.e., from 0 to 2a along the x-axis and from 0 to b along the y-axis, we find that

$$U_{bend} = \frac{ab}{4} \cdot \frac{\pi^4 n^4}{b^4} [f_1^2 (D_1 + 2D_3 + D_2) + f_2^2 (3D_1 + 2D_3 + 3D_2)]. \tag{39}$$

Substituting in (29) the expression for the stress function (36) and integrating over the same section of the shell surface, we find that

$$U_{sh} = hab \frac{p_0^2}{E_1^0} + \frac{hab}{2} \left[\frac{\pi^4 n^4}{b^4} (A_1 f_1^4 + A_2 f_2^4 + A_3 f_1^2 f_2^2) - \right.$$

$$\left. - \frac{\pi^2 n^2}{b^2 r} (A_4 f_2^2 f_2 + A_5 f_2^3) + \frac{1}{r^2} (A_6 f_1^2 + A_7 f_2^2)\right], \tag{40}$$

where

$$\left.\begin{aligned}
A_1 &= \frac{1}{64}\left(\frac{1}{K_2}+\frac{1}{K_1}\right);\\
A_2 &= \frac{17}{1\,024}\left(\frac{1}{K_2}+\frac{1}{K_1}\right)+\frac{1}{32}\left(\frac{1}{4x_2}+\frac{1}{x_1}+\frac{1}{4x_3}\right);\\
A_3 &= \frac{1}{32}\left(\frac{1}{K_2}+\frac{1}{K_1}\right)+\frac{1}{2x_1}+\frac{9}{8}\left(\frac{1}{x_4}+\frac{1}{x_5}\right);\\
A_4 &= \frac{1}{16K_2}+\frac{1}{x_1};\\
A_5 &= \frac{1}{16}\left(\frac{1}{K_2}+\frac{1}{x_1}\right);\\
A_6 &= \frac{1}{2x_1};\\
A_7 &= \frac{1}{16}\left(\frac{1}{K_2}+\frac{1}{2x_1}\right);
\end{aligned}\right\} \tag{41}$$

The work of the external forces /508

$$W = \Delta 2ap_0hb. \tag{42}$$

Here, Δ 2a is the contraction of a panel of length 2a.

Obviously,

$$\Delta 2a = -\int_0^{2a} \frac{\partial u}{\partial x}\,dx. \tag{43}$$

We have

$$\epsilon_x^0 = \frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \tag{44}$$

and

$$\epsilon_x^0 = \frac{1}{E_1^0}\left(\sigma_x^0 - \mu_1^0\sigma_y^0\right) = \frac{1}{E_1^0}\left(\frac{\partial^2 F}{\partial y^2} - \mu_1^0\frac{\partial^2 F}{\partial x^2}\right). \tag{45}$$

Equating (44) and (45), we obtain

$$\frac{\partial u}{\partial x} = \frac{1}{E_1^0}\left(\frac{\partial^2 F}{\partial y^2} - \mu_1^0\frac{\partial^2 F}{\partial x^2}\right) - \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2. \tag{46}$$

Then

$$W = -p_0 hb\int_0^{2a}\left[\frac{1}{E_1^0}\left(\frac{\partial^2 F}{\partial y^2} - \mu_1^0\frac{\partial^2 F}{\partial x^2}\right) - \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]dx. \tag{47}$$

Substituting (33) and (36) and integrating, we obtain

$$W = \frac{2p_0^2}{E_1^0}hab + p_0\pi^2 hn^2\left(\frac{1}{4}f_1^2 + \frac{3}{16}f_2^2\right). \tag{48}$$

Thus, the energy of the system

$$\begin{aligned}
\ni = \frac{hab}{2}&\left[\frac{\pi^4 n^4}{b^4}\left(A_1 f_1^4 + A_2 f_2^4 + A_3 f_1^2 f_2^2\right) - \right. \\
&\left. - \frac{\pi^2 n^2}{r^2}\left(A_4 f_1^2 f_2 + A_5 f_2^3\right) + \frac{1}{r^2}\left(A_6 f_1^2 + A_7 f_2^2\right)\right] + \\
&+ \frac{ab}{4}\cdot\frac{\pi^4 n^4}{b^4}\left[\left(D_1 + 2D_3 + D_2\right)f_1^2 + \left(3D_1 + 2D_3 + 3D_2\right)f_2^2\right] - \\
&- habK_1 p_0^2 - p_0\pi^2 hn^2\left(\frac{1}{4}f_1^2 + \frac{3}{16}f_2^2\right).
\end{aligned} \tag{49}$$

Conditions (31) yield a system of two equations:

$$\left.\begin{aligned} & f_1\left[\frac{\pi^2 n^2}{b^2}\left(4A_1 f_1^2 + 2A_3 f_2^2\right) - \frac{2A_4}{r} f_2 + \right. \\ & \left. + \frac{b^2}{\pi^2 n^2} \cdot \frac{2A_6}{r^2} + \frac{\pi^2 n^2}{b^2} \cdot \frac{D_1 + 2D_3 + D_2}{h} - p_0\right] = 0; \\ & \frac{\pi^2 n^2}{b^2}\left(4A_2 f_2^3 + 2A_3 f_1^2 f_2\right) - \frac{1}{r}\left(A_4 f_1^2 + 3A_5 f_2^2\right) + \\ & + \frac{b^2}{\pi^2 n^2} \cdot \frac{2A_7}{r^2} f_2 + \frac{\pi^2 n^2}{b^2} \cdot \frac{3D_1 + 2D_3 + 3D_2}{h} f_2 - \frac{3}{4} p_0 f_2 = 0. \end{aligned}\right\} \quad (50)$$

This system of cubic equations (50) defines all possible /509
types of equilibrium states of the shell. We will be primarily interested in those equilibrium states of the shell after buckling ($f_1 \neq 0$ and $f_2 \neq 0$) which correspond to the minimal value of the compressive stress p_0.

This minimum value of the stress at which the equilibrium state of the shell with large deflections is possible is called the "lower critical stress."

For simplicity in the analysis of equations (50), we will introduce dimensionless parameters for the stress, curvature, and elastic properties. The dimensionless parameters are the similarity criteria, and it is convenient to include in them both the experimental and theoretical relationships.

The dimensionless parameter for the compressive load

$$\bar{p}_0 = \frac{p_0 b^2}{\sqrt{E_1^0 E_2^0}\, h^2}. \quad (51)$$

The dimensionless parameter for the curvature of the shell

$$k = \frac{b^2}{rh} = \frac{\Theta b}{h}. \quad (52)$$

The dimensionless parameters for the elastic properties

$$\varphi = \frac{E_1^0}{E_2^0}\ ; \quad \Phi = \frac{E_1}{E_2}\ ; \quad \gamma = \sqrt{\frac{E_1\ E_2}{E_1^0\ E_2^0}}\ ;$$

$$\zeta = \frac{G^0}{\sqrt{E_1^0\ E_2^0}}\ , \quad Z = \frac{G_0}{\sqrt{E_1 E_2}}\ . \tag{53}$$

We will also use the following parameters:

$$\bar{\psi} = \frac{2G^0}{\sqrt{E_1^0\ E_2^0}}\left(1 + \sqrt{\mu_1^0\mu_2^0}\right); \quad \Psi = \frac{2G_0}{\sqrt{E_1E_2}}\left(1 + \sqrt{\mu_1\mu_2}\right). \tag{54}$$

They are convenient for comparison of the elastic properties of the orthotropic shell with the corresponding properties of the isotropic shell, since for the latter they are equal to unity.

Thus, the elastic properties of the orthotropic shell which is not homogeneous throughout the thickness are characterized by the following eight independent parameters (of which all except the first are dimensionless):

$$\sqrt{E_1^0E_2^0},\ \varphi,\ \Phi,\ \gamma,\ \zeta,\ Z,\ \sqrt{\mu_1^0\mu_2^0},\ \sqrt{\mu_1\mu_2}$$

or the other eight independent parameters

$$\sqrt{E_1^0E_2^0},\ \varphi,\ \Phi,\ \gamma,\ \psi,\ \Psi,\ \sqrt{\mu_1^0\mu_2^0},\ \sqrt{\mu_1\mu_2}.$$

The properties of an orthotropic shell which is uniform across the thickness are characterized by the four independent parameters

$$\sqrt{E_1E_2},\ \varphi,\ \zeta,\ \sqrt{\mu_1\mu_2} \text{ or } \sqrt{E_1E_2},\ \varphi,\ \psi,\ \sqrt{\mu_1\mu_2},$$

since for such a shell,

$$\sqrt{E_1^0E_2^0} = \sqrt{E_1E_2},\ \Phi = \varphi,\ \gamma = 1,\ Z = \zeta,\ \sqrt{\mu_1^0\mu_2^0} = \sqrt{\mu_1\mu_2}.$$

After transformation and the introduction of the dimensionless parameters, the system of equations (50) takes the form /510

$$\bar{p} = \pi^2n^2\theta + \frac{k^2}{\pi^2n^2d} + \pi^2n^2\ (4d_1 + 2d_3\alpha^2)\ \bar{f}_1^2 - 2d_4k\alpha\bar{f}_1; \tag{55}$$

$$\pi^2n^2\ (p_1\alpha^3 + p_2\alpha)\ \bar{f}_1^2 - k\ (d_4 + p_3\alpha^2)\ \bar{f}_1 + \frac{k^2}{\pi^2n^2}\,p_4\alpha + \pi^2n^2\theta_1\alpha = 0, \tag{56}$$

where

$$\bar{f}_1 = \frac{f_1}{h};\quad \bar{f}_2 = \frac{f_2}{h};\quad \alpha = \frac{f_2}{f_1}.$$

The constant coefficients in equations (55) and (56) have the following values:

$$\theta = \frac{\gamma}{12(1 - \mu_1\mu_2)}\left(\sqrt{\Phi} + A + \frac{1}{\sqrt{\Phi}}\right), \tag{57}$$

where

$$A = 2\sqrt{\mu_1\mu_2} + 4Z\ (1 - \mu_1\mu_2) = 2\left[\sqrt{\mu_1\mu_2} + \Psi\left(1 - \sqrt{\mu_1\mu_2}\right)\right], \tag{58}$$

$$\theta_1 = \frac{3\gamma}{16(1 - \mu_1\mu_2)}\left(\sqrt{\Phi} + \frac{A}{9} + \frac{1}{\sqrt{\Phi}}\right); \tag{59}$$

$$d = \sqrt{\varphi} + B + \frac{1}{\sqrt{\varphi}} , \tag{60}$$

where

$$B = \frac{1}{\zeta} - 2\sqrt{\mu_1^0\mu_2^0} = 2\left(\frac{1 + \sqrt{\mu_1^0\mu_2^0}}{\psi} - \sqrt{\mu_1^0\mu_2^0}\right); \tag{61}$$

$$d_1 = \frac{1}{64}\left(\sqrt{\varphi} + \frac{1}{\sqrt{\varphi}}\right); \tag{62}$$

$$d_3 = 2d_1 + \frac{1}{2d} + \frac{9}{8}\left(\frac{1}{\sqrt{\varphi} + 9B + \frac{81}{\sqrt{\varphi}}} + \frac{1}{81\sqrt{\varphi} + 9B + \frac{1}{\sqrt{\varphi}}}\right) \tag{63}$$

$$d_4 = \frac{1}{16\sqrt{\varphi}} + \frac{1}{d} ; \tag{64}$$

$$p_1 = 4d_2 - \frac{3}{2} d_3; \tag{65}$$

$$p_2 = 2d_3 - 3d_1; \tag{66}$$

$$p_3 = 3d_5 - \frac{3}{2} d_4; \tag{67}$$

$$p_4 = 2d_7 - \frac{3}{4d} , \tag{68}$$

where

$$d_2 = \frac{17}{16} d_1 + \frac{1}{128} \left(\frac{1}{\sqrt{\varphi} + 4B + \frac{16}{\sqrt{\varphi}}} + \right.$$

$$\left. + \frac{4}{d} + \frac{1}{16 \sqrt{\varphi} + 4B + \frac{1}{\sqrt{\varphi}}} \right); \qquad (69) /511$$

$$d_5 = \frac{1}{16} \left(\frac{1}{\sqrt{\varphi}} + \frac{1}{d} \right); \qquad (70)$$

$$d_7 = \frac{1}{16} \left(\frac{1}{\sqrt{\varphi}} + \frac{1}{2d} \right). \qquad (71)$$

In the particular case of the isotropic shell, $\varphi = \Phi = \gamma = \Psi = 1$ and $\mu_1\mu_2 = \mu = 0.3$, the system of equations (55) and (56) has the form

$$\bar{p} = 3.62n^2 + \frac{k^2}{4\pi^2 n^2} + \frac{\pi^2}{8} n^2 (1 + 3.36\alpha^2) \bar{f}_1^2 - 0.625k\alpha\bar{f}_1; \qquad (72)$$

$$n^2 (1.465\alpha^3 - 3.22\alpha) \bar{f}_1^2 - \frac{5}{16} k \left(\frac{3}{4} \alpha^2 - 1 \right) \bar{f}_1 +$$

$$+ \frac{3}{632} \cdot \frac{k^2}{n^2} \alpha - 4.53n^2\alpha = 0, \qquad (73)$$

i.e., it coincides with the system of equations obtained by A. S. Vol'mir (Ref. 3).*

"Upper critical stress," p_u, is the term given to the value of the stress at which two forms of elastic equilibrium are equally possible: the initial-cylindrical and the buckled, infinitely close to

*Reference 3 contains misprints in certain coefficients.

the initial form; i.e., in determining p_u, the stability "in the small" is considered. Setting $\bar{f}_1 = 0$ and $\bar{f}_2 = 0$ in (55), we obtain the equation defining the equilibrium state of the system for infinitesimal deflections with differing numbers of halfwaves:

$$\bar{p}_0 = \pi^2 n^2 \theta + \frac{k^2}{\pi^2 n^2 d} . \tag{74}$$

Using equations (55) and (56), we can construct the variation of the compressive stress which will provide for the equilibrium state of the shell as a function of the deflection.

To do this, we assume a value of the curvature parameter k and determine from equation (74) the lowest value of $\bar{p}_0$ and the corresponding value of n; these values will be $\bar{p}_u$, the upper critical stress, and n_u, the corresponding number of halfwaves.

Then, for values of n equal to and less than n_u, we determine $\bar{p}$, assuming for each value of n a series of values of α. Substituting α in (56), we obtain a quadratic equation in f_1 and find its roots (real and positive). Having α and the values of the roots, from equation (55) we obtain one or two values of $\bar{p}$ corresponding to one or two roots $\bar{f}_1^{(1)}$, $\bar{f}_1^{(2)}$. The values of α must be assumed so that equation (56) has real roots.

Similar calculations were carried out for the isotropic shell. The plot of the relation $\bar{p} = \varphi_1 (\bar{f}_1)$ for k = 140 is presented in Fig. 2. /512

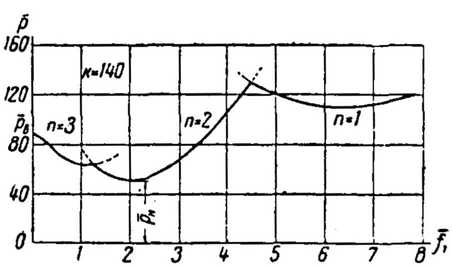

Fig. 2

An analysis of the relation $\bar{p} = \varphi_1 (\bar{f}_1)$ was made by A. S. Vol'mir in Ref. 3. We will be interested here only in the lowest balue of $\bar{p}$ -- the lower critical stress $\bar{p}_1$.

3. THE CRITICAL STRESS FOR A COMPRESSED CIRCULAR CYLINDRICAL ORTHOTROPIC SHELL

We will now derive the formula for the upper critical stress. Equation (74) gives a family of parabolas in the $\bar{p}_0$, k-coordinates with parameter n. We will find the envelope of this family. We will obtain the equation of the envelope if we exclude the parameter n from the system of equations

$$F_1 = 0 \text{ and } \frac{\partial F_1}{\partial n} = 0,$$

where

$$F_1 (k, \bar{p}_0, n) = \pi^2 n^2 \theta + \frac{k^2}{\pi^2 n^2 d} - \bar{p}_0.$$

As a result, we find

$$\bar{p}_u = 2k \sqrt{\frac{\theta}{d}} \tag{75}$$

which is the equation of a line passing through the coordinate origin. We will denote the slope of this line by χ_u, i.e.,

$$\bar{p}_u = \chi_u k, \tag{76}$$

where

$$\chi_u = 2 \sqrt{\frac{\theta}{d}} . \tag{77}$$

The quantity χ_u may be obtained by another method. We find the minimum of the function

$$\chi = \frac{\bar{p}_0}{k} = \frac{\pi^2 n^2 \theta}{k} + \frac{k}{\pi^2 n^2 d}. \tag{78}$$

We equate its derivative to zero:

$$\frac{\partial \chi}{\partial k} = - \frac{\pi^2 n^2 \theta}{k^2} + \frac{1}{\pi^2 n^2 d} = 0. \tag{79}$$

This yields

$$\frac{k}{n^2} = \pi^2 \sqrt{\theta d}. \tag{80}$$

Substituting this expression in (78), we obtain (77).

Since $n \geq 1$ (the number of halfwaves cannot be less than unity), from (80) we obtain the condition for which equation (77) is valid:

$$k \geq \pi^2 \sqrt{\theta d} = k_{1_u}. \tag{81}$$

For a shell of small curvature $(0 \leq k \leq k_{1u})$ $n = 1$, and the upper critical stress parameter is determined from (74) in the form /513

$$\bar{p}_u = \pi^2 \theta + \frac{k^2}{\pi^2 d}. \tag{82}$$

From (51), we have

$$p_u = \bar{p}_u \sqrt{E_1^0 E_2^0} \frac{h^2}{b^2}. \tag{83}$$

Substituting (76) and the expression for k (52), we obtain

$$\text{when } k \geq k_{1_u}$$

$$p_u = \chi_u \sqrt{E_1^0 E_2^0}\, \frac{h}{r}. \tag{84}$$

For the small-curvature shell, substituting (82) in (83), we obtain

$$\text{when } 0 \leq k \leq k_{1_u}$$

$$p_u = \pi^2\theta \sqrt{E_1^0 E_2^0}\, \frac{h^2}{b^2} + \frac{\sqrt{E_1^0 E_2^0}}{\pi^2 d} \cdot \frac{b^2}{r^2}\,. \tag{85}$$

Equations (84) and (85) are valid for $a \geq b$. They may be used for any ratio of the sides of the panel, a/b, greater than unity, since the critical stress of the simply supported shell does not vary significantly with the aspect ratio of the panel for $a/b > 1$.

For the isotropic shell with $\mu = 0.3$, $\chi_u = 0.6$, and consequently, we obtain the familiar formula

$$\text{when } k \geq 12 \quad p_u = 0.6\, E\, \frac{h}{r}\,. \tag{86}$$

Using the method described at the end of Section 2, we can calculate $\overline{p}_1$. However, this method is not suitable in practice, since it requires extensive computational labor which must be repeated for any change in any of the elastic parameters.

Therefore, we shall turn to equations (55) and (56), which may now be written in the form

$$\chi = \theta\xi + \frac{1}{\xi d} + 4d_1\xi\overline{f}_1^2 + 2d_3\xi\overline{f}_2^2 - 2d_4\overline{f}_2; \tag{87}$$

$$p_1\xi\overline{f}_2^3 + p_2\xi\overline{f}_1^2\,\overline{f}_2 + \theta_1\xi\overline{f}_2 + \frac{p_4}{\xi}\,\overline{f}_2 - d_4\overline{f}_1^2 - p_3\overline{f}_2^2 = 0, \tag{88}$$

where

$$\xi = \frac{\pi^2 n^2}{k} . \qquad (89)$$

In brief form, the set of equations (87) and (88) is expressed as:

$$\chi = F_2 (\bar{f}_1, \bar{f}_2, \xi);$$

$$\varphi_1 (\bar{f}_1, \bar{f}_2, \xi) = 0.$$

We will find the relative minimum of the function F_2. To do this, we construct the function $\Phi_1 (\bar{f}_1, \bar{f}_2, \xi, \lambda) = F_2 (\bar{f}_1, \bar{f}_2, \xi) + \lambda \varphi_1 (\bar{f}_1, \bar{f}_2, \xi)$, where λ is the Langrangian multiplier (Ref. 4).

The necessary conditions for the extremum of the function Φ_1 yield a set of equations in the unknowns $\bar{f}_1$, $\bar{f}_2$, ξ, and λ. These equations have the form /514

$$\varphi_1 = 0; \quad \frac{\partial \Phi_1}{\partial f_1} = 0; \quad \frac{\partial \Phi_1}{\partial \bar{f}_1} = 0; \quad \frac{\partial \Phi_1}{\partial \xi} = 0. \qquad (90)$$

The set of solutions $(\bar{f}_{1_1}, \bar{f}_{2_1}, \xi_1, \lambda_1)$ satisfying these equations can give the minimum for the function F_2.

We have

$$\Phi_1 (f_1, f_2, \xi, \lambda) = \theta\xi + \frac{1}{\xi d} + 4d_1\xi\bar{f}_1^2 + 2d_3\xi\bar{f}_2^2 - 2d_4\bar{f}_2 +$$

$$+ \lambda (p_1\xi\bar{f}_2^3 + p_2\xi\bar{f}_1^2 \bar{f}_2 + \theta_1\xi\bar{f}_2 + \frac{p_4}{\xi} \bar{f}_2 - d_4 \bar{f}_1^2 - p_3 \bar{f}_1^2).$$

The set of equations (90) becomes

$$p_1\xi\bar{f}_2^3 + p_2\xi\bar{f}_1^2 \bar{f}_2 + \theta_1\xi\bar{f}_2 + \frac{p_4}{\xi} \bar{f}_2 - d_4 \bar{f}_1^2 - p_3\bar{f}_2^2 = 0; \qquad (91)$$

$$8d_1\xi\bar{f}_1 + \lambda\ (2p_2\xi\bar{f}_1\bar{f}_2 - 2d_4\bar{f}_1) = 0; \tag{92}$$

$$4d_3\xi\bar{f}_2 - 2d_4 + \lambda\ (3p_1\xi\bar{f}_2^2 + p_2\xi\bar{f}_1^2 + \theta_1\xi + \frac{p_4}{\xi} - 2p_3\bar{f}_2) = 0; \tag{93}$$

$$\theta - \frac{1}{d\xi^2} + 4d_1\bar{f}_1^2 + 2d_3\bar{f}_2^2 +$$

$$+ \lambda\ (p_1\bar{f}_1^3 + p_2\bar{f}_1^2\ \bar{f}_2 + \theta_1\bar{f}_2 - \frac{p_4}{\xi^2}\ \bar{f}_2) = 0. \tag{94}$$

We will solve the set of equations (91)-(94) as follows. From (92), we will determine λ and substitute its value into (93) and (94). Then, we will make the change of variables

$$\xi\bar{f}_1 = x_1; \quad \xi\bar{f}_2 = y_1; \quad \xi^2 = z. \tag{95}$$

The set of equations takes the form

$$p_1y_1^3 + p_2x_1^2y_1 - d_4x_1^2 - p_3y_1^2 + p_4y_1 + \theta_1y_1z_1 = 0; \tag{96}$$

$$a_1y_1^2 + a_5x_1^2 + a_2y_1 + a_3z_1 + a_4 = 0; \tag{97}$$

$$b_1y_1^3 + b_2x_1^2 + b_3y_1^2 + b_4y_1 + b_5y_1z_1 + b_6z_1 - b_7 = 0, \tag{98}$$

where

$$\left.\begin{array}{lll}
a_1 = 3d_1p_1 - d_3p_2; & a_5 = d_1p_2; & b_4 = \frac{p_2}{d} - 4d_1p_4; \\
a_2 = d_3d_4 + \frac{1}{2}\ d_4p_2 - 2d_1p_3; & b_1 = 4d_1p_1 - 2d_3p_2; & b_5 = 4d_1\theta_1 - p_2\theta; \\
a_3 = d_1\theta; & b_2 = 4d_1d_4; & b_6 = d_4\theta; \\
a_4 = d_1p_4 - \frac{1}{2}\ d_4^2; & b_3 = 2d_3d_4; & b_7 = \frac{d4}{d}.
\end{array}\right\} \tag{99}$$

From (97), we determine

$$x_1^2 = -\frac{1}{a_5}(a_1y_1^2 + a_2y_1 + a_3z_1 + a_4). \tag{100}$$

Substituting the value of x_1^2 in (96) and (98), we obtain /515

$$c_1y_1^3 + c_2y_1^2 + c_3y_1 + c_4\,y_1z_1 + c_5z_1 + c_6 = 0; \tag{101}$$

$$b_1y_1^3 + c_7y_1^2 + c_8y_1 + b_5y_1z_1 + c_9z_1 - c_{10} = 0, \tag{102}$$

where

$$\left.\begin{array}{lll} c_1 = p_1 - \frac{p_2}{a_5}a_1; & c_6 = \frac{d_4}{a_5}a_4; & c_3 = \frac{d_4}{a_5}a_2 - \frac{p_2}{a_5}a_4 + p_4; \\ c_2 = \frac{d_4}{a_5}a_1 - \frac{p_2}{a_5}a_2 - p_3; & c_7 = -\frac{b_2}{a_5}a_1 + b_3; & c_9 = b_6 - \frac{b_2}{a_5}a_3; \\ c_4 = \theta_1 - \frac{p_2}{a_5}a_3; & c_8 = b_4\frac{b_2}{a_5}a_1; & c_{10} = b_7 + b_2\frac{a_4}{a_5}; \\ c_5 = \frac{d_4}{a_5}a_3. & & \end{array}\right] \tag{103}$$

Multiplying equations (101) and (102) by b_1 and c_1 and subtracting one from the other, we obtain

$$z_1 = -\frac{A_1y_1^2 + A_2y_1 + A_5}{A_3y_1 + A_4} \tag{104}$$

where

$$\left.\begin{array}{ll} A_1 = c_2b_1 - c_1c_7; & A_4 = b_1c_5 - c_1c_9; \\ A_2 = b_1c_3 - c_1c_8; & A_5 = b_1c_6 - c_1c_{10}. \\ A_3 = b_1c_4 - b_5c_1; & \end{array}\right\} \tag{105}$$

Substituting (104) in (102), we obtain the equation of the fourth degree in the unknown y_1:

$$B_1 y_1^4 + B_2 y_1^3 + B_3 y_1^2 + B_4 y_1 + B_5 = 0, \tag{106}$$

where

$$\left.\begin{aligned} &B_1 = b_1 A_3; && B_4 = c_8 A_4 - b_5 A_5 - c_9 A_2 - c_{10} A_3; \\ &B_2 = b_1 A_4 + c_7 A_3 - b_5 A_1; && B_5 = - c_9 A_5 - c_{10} A_4 \\ &B_3 = c_7 A_4 + c_8 A_3 - b_5 A_2 - c_9 A_1; && \end{aligned}\right\} \tag{107}$$

are the coefficients, expressed in terms of the elastic parameters.

Equation (106) may be solved by any of the approximate methods [for example, the Sturm method (Ref. 5)].

Only the real positive roots are of significance. Substituting the values of the roots in (104), we obtain z_1, and substituting y_1 and z_1 in (100), we obtain x_1. Changing back to the original variables by using (95), we find $\bar{f}_1$, $\bar{f}_2$, and ξ, and then we substitute these values in (87) and obtain χ_1.

We calculate the lower critical stress by the formula

$$p_1 = \chi_1 \sqrt{E_1^0 E_2^0 \frac{h}{r}} \tag{108}$$

for $k \geq k_1$,

where

$$k_1 = \frac{\pi^2}{\xi_1} . \tag{109}$$

We obtain the value of k_1 from (89) if we set $n = 1$. /516

We can make use of another method for obtaining the relation $\bar{p}_1 = f(k)$, which is also suitable for $0 \le k \le k_1$.

Let k and n take fixed values (where the n are whole numbers).

We rewrite (55) and (56) as follows:

$$\bar{p} = \pi^2 n^2 \theta + \frac{k^2}{\pi^2 n^2 d} + 4\pi^2 d_1 n^2 \bar{f}_1^2 + 2\pi^2 d_3 n^2 \bar{f}_2^2 - 2d_4 k \bar{f}_2; \tag{110}$$

$$\pi^2 n^2 (p_1 \bar{f}_2^3 + p_2 \bar{f}_1^2 \bar{f}_2) - k (d_4 \bar{f}_1^2 + p_3 \bar{f}_2^2) + \\ + \frac{p_4}{\pi^2 n^2} k^2 \bar{f}_2 + \pi^2 n^2 \theta \bar{f}_2 = 0. \tag{111}$$

This system may be written in short form as:

$$\bar{p} = F_3 (\bar{f}_1, \bar{f}_2);$$

$$\varphi_2 (\bar{f}_1, \bar{f}_2) = 0.$$

We will now determine the relative minimum of the function F_3. To do this, we construct $\Phi_2 = F_3 + \lambda \varphi_2$. The extremum of the function F_3 is determined by the set of solutions $(\bar{f}_{1_1}, \bar{f}_{2_1}, \lambda_1)$, which is obtained from the equations

$$\varphi_2 = 0; \quad \frac{\partial \Phi_2}{\partial \bar{f}_1} = 0; \quad \frac{\partial \Phi_2}{\partial \bar{f}_2} = 0, \tag{112}$$

i.e.,

$$\pi^2 n^2 p_1 \bar{f}_2^3 + \pi^2 n^2 p_2 \bar{f}_1^2 \bar{f}_2 - d_4 k \bar{f}_1^2 - p_3 k \bar{f}_2^2 + \frac{p_4}{\pi^2 n^2} k^2 \bar{f}_2 + \pi^2 n^2 \theta_1 \bar{f}_2 = 0; \tag{113}$$

$$8\pi^2 n^2 d_1 \bar{f}_1 + \lambda (2\pi^2 n^2 p_2 \bar{f}_1 \bar{f}_2 - 2d_4 k \bar{f}_1) = 0; \tag{114}$$

$$4\pi^2 n^2 d_3 \bar{f}_2 - 2d_4 k + \lambda\,(3\pi^2 n^2 p_1 \bar{f}_2^2 +$$

$$+ \pi^2 n^2 p_2 \bar{f}_1^2 - 2p_3 k \bar{f}_2 + \frac{p_4}{\pi^2 n^2} k^2 + \pi^2 n^2 \theta_1) = 0. \qquad (115)$$

Excluding λ from this set, we obtain

$$e_1 \bar{f}_2^3 + e_2 \bar{f}_1^2 \bar{f}_2 - e_3 \bar{f}_1^2 - e_4 \bar{f}_2^2 + e_5 \bar{f}_2 = 0; \qquad (116)$$

$$S_1 \bar{f}_2^2 + S_2 \bar{f}_2 + S_3 = 0, \qquad (117)$$

where

$$\left.\begin{aligned} e_1 &= \pi^2 p_1 n^2; & e_5 &= \frac{p_4}{\pi^2 n^2} k^2 + \pi^2 \theta_1 n^2; \\ e_2 &= \pi^2 p_2 n^2; & S_1 &= 2\pi^4\,(3d_1 p_1 + d_1 p_2 - d_3 p_2)\,n^4; \\ e_3 &= d_4 k; & S_2 &= \pi^2\,(2d_3 d_4 + d_4 p_2 - 4d_1 p_3)\,kn^2; \\ e_4 &= p_3 k; & S_3 &= (2d_1 p_4 - d_4^2)\,k^2 + 2\pi^4 d_1 \theta_1 n^4. \end{aligned}\right\} \qquad (118)$$

Solving the quadratic equation (117), we obtain two values of $\bar{f}_2$, and from (116), we have

$$\bar{f}_1^2 = \frac{e_1 \bar{f}_2^2 - e_4 \bar{f}_2 + e_5}{e_3 - e_2 \bar{f}_2}\,\bar{f}_2. \qquad (119)$$

We must use only that positive root of equation (117) which does not make the right side of (119) a negative quantity. /517

In order to obtain the values of the lower critical stress for $0 \le k \le k_1$, it is necessary to set $n = 1$, calculate the coefficients (118) for a given k, determine the values of $\bar{f}_1$, $\bar{f}_2$, and substitute them in (110).

Performing the calculation for a series of values of k, we obtain the relation $\overline{p}_1 = f_1(k)$ for n = 1.

By the same method, we can determine $p_1 = f_n(k)$ for n = 2, 3, ... Here, the increasing value of k corresponds to the larger values of n ($k \geq k_1$).

Figure 3 presents a plot of $\overline{p}_1 = f_n(k)$ for the isotropic shell calculated by this method. The values of $\overline{p}_u$ are also shown.

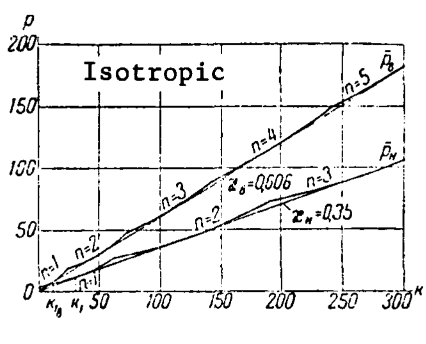

Fig. 3

As may be seen from the graph, the curves of the lower critical stress parameter for a shell of large curvature approximate a straight line passing through the coordinate origin. The slope of this line $\chi_1 = 0.35$*. A different value for χ_1 was obtained in Ref. 3 as a result of inaccuracies in the coefficients of the set of equations (72) and (73).

The magnitude of the slope of the straight line, which is the envelope of the family of curves $\overline{p}_1 = f_n(k)$ for various n, may be determined analytically, as indicated above [system (90)].

Calculations were performed for certain combinations of elastic parameters of an orthotropic shell; the results are presented in the Table.

*This value is too high and will be refined in Section 5.

λ	φ	Φ	ψ	Ψ	$\sqrt{\mu_1^0 \mu_2^0}$	$\sqrt{\mu_1 \mu_2}$	χ_u	χ_1	Shell
1	1	-	1	-	-	0.3	0.606	0.350	Isotropic
1	2	-	1	-	-	0.3	0.605	0.412	Uniform across thickness, orthotropic
1	3	-	1	-	-	0.3	0.605	0.449	Same
1	1	-	0.5	-	-	0.3	0.428	0.320	Same
2.41	1.72	9.97	0.718	0.265	0.244	0.101	0.817	0.645	Dural specimens with longitudinal stiffeners

We can draw certain qualitative conclusions from these results concerning the effect of the elastic parameters on the parameters $\bar{p}_u$ and $\bar{p}_1$. For example, with increasing μ or decreasing ψ, the difference between the upper and lower critical stresses diminishes. At the same time, it is clear that there is a reduction of the possibility of "snap" occurring during the buckling.

4. SEQUENCE OF THE CALCULATION OF THE CRITICAL STRESS FOR COMPRESSION OF THE CIRCULAR CYLINDRICAL ORTHOTROPIC SHELL /518

1. Determine the elastic parameters of the shell material experimentally or theoretically (see Ref. 6).

2. Calculate $\sqrt{E_1^0 E_2^0}$ and the dimensionless elastic parameters (53), (54).

3. Calculate the coefficients (57)-(71).

4. Determine the coeffcients (118).

5. Assume the number n: for small k, take n = 1, for large k assume several values n = 1, 2, 3, ... Do the same in the case in

which the value of k_1 is unknown and it is impossible to establish whether the value of k is greater or less than the value of k_1 [to find k_1, it is necessary to solve the set of equations (87), (88), and this requires a great amount of time].

6. Solve the system (116), (117), and substitute the solution in (110). For small k, formula (110) gives the value of $\bar{p}_1$. For large k, it is necessary to carry out these calculations for several values of n and take the lowest value -- which will be $\bar{p}_1$.

If it is required that we find the quantity χ_1, then, after point 3, we should:

4. Determine the coefficients (99), (103), (105), (107).

5. Solve equation (106) approximately and find χ_1 from formula (87).

The magnitude of χ_1 can also be found as the slope of the envelope of the family of curves $\bar{p}_1 - f_n(k)$ with the parameter n.

To determine the upper critical stress parameter, it is necessary, after step 2, to

3. Calculate the coefficients (57), (58), (60), (61).

4. Determine $\bar{p}_u$ from equations (77), (84), (85).

An example of the calculation of $\bar{p}_1$ for a specimen which has been tested experimentally (Ref. 7) follows.

The specimen consisted of a thin sheet wrapped into a circular cylinder and reinforced with closely spaced stiffeners along the generator on both the inside and outside. Frames were used at the edges and the ends, which corresponded approximately to the boundary conditions used in the theoretical consideration of the problem.

The cross-section of the specimen perpendicular to the generator is shown in Fig. 4. The stiffener spacing is 2.5 cm, h = 0.077 cm; the section of each stiffener is $a_s h_1 = 0.9 \cdot 0.077 = 0.0693$ cm^2. The width of the specimen along the arc is b = 20 cm, the radius of curvature is r = 18.7 cm, and the length a = 40 cm. The sample is made of duraluminum.

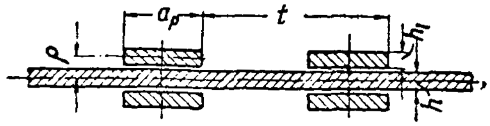

Fig. 4

1. We will refer all the parameters of the shell stiffeners to the thickness of the sheet h. The elastic modulus along the arc is equal to the modulus of the material $E_2^0 = E = 7.5 \cdot 10^5$ kg/cm^2. The elastic modulus E_1^0 along the generator, from the condition of equality of the forces on a unit length of the section, must be increased in the ratio f_1/f_2, where f_1 is the area per unit length of the section perpendicular to the generator and f_2 is the area per unit length of the section parallel to the generator: /519

$$\frac{f_1}{f_2} = 1 + 2\frac{a_p}{t} = 1.72,$$

i.e.,

$$E_1^0 = E\frac{f_1}{f_2} = 12.9 \cdot 10^5 \text{ kg/cm}^2; \quad G^0 = G = 2.84 \cdot 10^5 \text{ kg/cm}^2;$$

$$\mu_1^0 = \mu = 0.32; \quad \mu_2^0 = \mu_1^0 \frac{E_2^0}{E_1^0} = 0.186;$$

$$\sqrt{\mu_1\mu_2} = \sqrt{0.32} \cdot 0.186 = 0.244.$$

The cylindrical stiffness of the reinforced shell (Ref. 6),

$$D_1 = \frac{Eh^3}{12\,(1-\mu^2)} + \frac{E'J}{t};$$

$$D_2 = \frac{Eh^3}{12\ (1 - \mu^2)}\ ;$$

$$D_3 = \frac{Eh^3}{12\ (1 - \mu^2)}\ ,$$

where E, E' - moduli of the shell and the stiffener (E' = E);
J - the moment of inertia of the area of the section of the stiffener relative to the middle surface,

$$J = 2\left(a_p h\rho^2 + \frac{a_p h_1^3}{12}\right) = 2\ (9 \cdot 0.077 \cdot 0.079^2 +$$

$$+ \frac{9 \cdot 0.077^3}{12} = 0.000931\ \mathrm{cm}^4.$$

Here, ρ is the distance from the center of the area of the stiffener to the middle surface (ρ = 0.079 cm);

$$\frac{E'J}{t} = 280\ \mathrm{kg/cm};$$

$$D_2 = 31.2\ \mathrm{kg/cm};\quad D_1 = 311\ \mathrm{kg/cm};\quad D_3 = 31.2\ \mathrm{kg/cm};$$

$$\frac{D_1}{D_2} = \frac{E_1}{E_2} = \Phi;\ \frac{D_1}{D_2} = \frac{311}{31.2} = 9.97;$$

$$E_2 = 7.5 \cdot 10^5\ \mathrm{kg/cm}^2;\quad E_1 = \Phi E_2 = 74.8 \cdot 10^5\ \mathrm{kg/cm}^2;$$

$$G_0 = G = 2.84 \cdot 10^5\ \mathrm{kg/cm}^2;\quad \mu_1 = \mu = 0.32;$$

$$\mu_2 = \mu_1 \frac{D_2}{D_1} = 0.032;\quad \sqrt{\mu_1\mu_2} = \sqrt{0.032} \cdot 0.32 = 0.101;$$

$$2.\ \sqrt{E_1^0 E_2^0} = 9.45 \cdot 10^5\ \mathrm{kg/cm}^2;$$

the dimensionless parameters:

$$\varphi = 1.72; \quad \psi = 0.718; \quad \sqrt{\mu_1^0\mu_2^0} = 0.244; \quad \gamma = 2.41;$$

$$\Phi = 9.97; \quad \Psi = 0.265; \quad \sqrt{\mu_1\mu_2} = 0.101.$$

3. The coefficients of the system (55), (56) have the following values: /520

$$\begin{array}{lll} A = 0.678; & d_2 = 0.0411; & d = 5.05; \\ B = 2.978; & d_3 = 0.185; & p_1 = -0.113; \\ \theta = 0.843; & d_4 = 0.2456; & p_2 = 0.272; \\ \theta_2 = 1.63; & d_5 = 0.06; & p_3 = 0.188; \\ d_1 = 0.0324; & d_7 = 0.0538; & p_4 = -0.0407. \end{array}$$

4. The coefficients of (118) in the given case will be

$$\begin{array}{ll} e_1 = -1.112n^2; & S_1 = -10.22n^4; \\ e_2 = 2.686n^2; & S_2 = 1.796kn^2; \\ e_3 = 0.246k; & S_3 = 10.3n^4 - 0.0628k^2; \\ e_4 = -0.188\,k; & \\ e_5 = 16.09n^2 - 0.00412\,\dfrac{k^2}{n^2}. & \end{array}$$

5. The curvature parameter is equal to $k = b^2/rh = 284$. The calculation was carried out for $n = 1, 2, 3$; $n = 2$ yielded the lowest $\bar{p}$. We will show the calculation only for $n = 2$.

For k = 284 and n = 2,

$$e_1 = -4.45; \quad e_5 = -18.8;$$
$$e_2 = 10.74; \quad S_1 = -163.3;$$
$$e_3 = 69.7; \quad S_2 = 2\ 040$$
$$e_4 = -53.4; \quad S_3 = -4\ 900.$$

Equation (117)

$$-163.3\ \bar{f}_2^2 + 2\ 040\bar{f}_2 - 4\ 900 = 0$$

has the roots $\bar{f}_2 = 3.24;\ 9.26.$

From equation (119), we find

$$\bar{f}_1^2 = \frac{-4.45 \cdot 10.5 + 53.4 \cdot 3.24 - 18.8}{69.7 - 10.74 \cdot 3.24}\ 3 \cdot 24 = 10.$$

The root $\bar{f}_{2_2} = 9.26$ gives the minimum value of $\bar{f}_1$ and therefore we disregard it.

According to formula (110) we obtain

$$\bar{p}_1 = 8.315 \cdot 4 + 0.02005\ \frac{284^2}{4} + 4 \cdot 1.279 \cdot 10 + 4 \cdot 3.65 \cdot 10.5 -$$

$$-0.4912 \cdot 284 \cdot 3.24 = 189.$$

The calculations were performed in a similar fashion for several values of k. These calculations established that in this case, the slope of the tangent to the curve $\bar{p}_1 = f(k)$ (i.e., χ_1) is equal to 0.645. Then, $\bar{p}_1 = \chi_1 \cdot k = 0.645 \cdot 284 = 183.$* The lower critical stress

*It is possible not to calculate χ_1 and limit oneself to the value obtained above with the formula (110): $\bar{p}_1 = 189.$

$$p_1 = \chi_1 \sqrt{E_1^0 E_2^0}\, \frac{h}{r} = p_1 \sqrt{E_1^0 E_2^0}\, \frac{h^2}{b^2} = 183 \cdot 9.45 \cdot 10^5\, \frac{0.077^2}{20^2} = 2\ 490\ \text{kg/cm}^2.$$

In the experimental testing of this specimen, the critical /521 stress, referred to the shell thickness, was found to be $p_e = 2.180$ kg/cm^2; i.e., the critical loads, $P_1 = p_1bh$, from the theoretical computation and from experiment were $P_1 = 3.840$ kg and $P_e = 3.360$ kg, $P_e/P_1 = 0.875$. This ratio between the theoretical and experimental data is acceptable in stability calculations.

The magnitude of $\chi_1 = 0.35$ obtained for the isotropic shell (see Fig. 3 and Table) is also sufficiently close to the average experimental data for the isotropic shell (Ref. 3), which give $\chi = 0.3$; i.e., $p_1 = 0.3$ Eh/r.

However, it must be kept in mind that the shell may buckle at stresses below the average values. Therefore, in design calculations, it is necessary to use values somewhat below the theoretical values of the stresses.

5. REFINED SOLUTIONS

We will use (32) as an expression for the deflection at buckling:

$$w = f_1 \sin\frac{n\pi x}{a} \sin\frac{n\pi y}{b} + f_2 \sin^2\frac{n\pi x}{a} \sin^2\frac{n\pi y}{b}\ .$$

We will denote $a/b = \epsilon$ - the ratio of the length of a halfwave along the generator to the length of a halfwave along the arc. We will take the quantity ϵ to be fixed, not varying with changes in f_1 and f_2. The solution performed by the method described in Section 2 gives the following results: the system of equations (55) and (56) remains unchanged, but its coefficients θ, θ_1, d, d_1 will have the form

$$\theta = \frac{\gamma}{12\ (1 - \mu_1\mu_2)} \left(\frac{\sqrt{\Phi}}{\epsilon^2} + A + \frac{\epsilon^2}{\sqrt{\Phi}}\right); \qquad (57')$$

$$\theta_1 = \frac{3\gamma}{16\ (1 - \mu_1\mu_2)}\left(\frac{\sqrt{\Phi}}{\epsilon^2} + \frac{A}{9} + \frac{\epsilon^2}{\sqrt{\Phi}}\right); \qquad (59')$$

$$d = \frac{\sqrt{\varphi}}{\epsilon^2} + B + \frac{\epsilon^2}{\sqrt{\varphi}}\ ; \qquad (60')$$

$$d_1 = \frac{1}{64}\ \frac{\epsilon^2}{\sqrt{\varphi}} + \frac{\sqrt{\varphi}}{\epsilon^2}\ ; \qquad (62')$$

$$d_3 = 2d_1 + \frac{1}{2d} + \frac{9}{8}\left(\frac{1}{\frac{\sqrt{\varphi}}{\epsilon^2} + 9B + \frac{81\ \epsilon^2}{\sqrt{\varphi}}} + \frac{1}{\frac{81\sqrt{\varphi}}{\epsilon^2} + 9B + \frac{\epsilon^2}{\sqrt{\varphi}}}\right); \qquad (63')$$

$$d_4 = \frac{\epsilon^2}{16\ \sqrt{\varphi}} + \frac{1}{d}\ ; \qquad (64')$$

$$d_2 = \frac{17}{16}\ d + \frac{1}{128}\left(\frac{1}{\frac{\sqrt{\varphi}}{\epsilon^2} + 4B + \frac{16\ \epsilon^2}{\sqrt{\varphi}}} + \frac{4}{d} + \frac{1}{\frac{16\sqrt{\varphi}}{\epsilon^2} + 4B + \frac{\epsilon^2}{\sqrt{\varphi}}}\right); \qquad (69')$$

$$d_5 = \frac{1}{16}\ \left(\frac{\epsilon^2}{\sqrt{\varphi}} + \frac{1}{d}\right); \qquad (70')$$ /522

$$d_7 = \frac{1}{16}\left(\frac{\epsilon^2}{\sqrt{\varphi}} + \frac{1}{2d}\right). \qquad (71')$$

The coefficients p_1, p_2, p_3, p_4 are determined by the same equations (65), (66), (67), and (68) but with consideration for the new values of the coefficients d_1, d_2, d_3, d_4, d_5, d_7, d.

To determine the values of the lower critical stress parameter $\bar{p}_1$, we will use the method presented in Section 3. In doing this, it is necessary in all cases to take the values of the coefficients of the equations from this Section. Thus, assuming a definite value of ϵ and using the sequence of calculation shown in Section 4, we can, for any circular cylindrical orthotropic shell, determine $\bar{p}_1$ for various k for the number n = 1, 2, 3, ... Assuming a series of values of ϵ and performing the calculations for each of them, we can establish that $\epsilon = \epsilon_0$ for every value of k for which $\bar{p}_1$ is a minimum. This value of $\bar{p}_1$ is then used for the lower critical stress parameter in the given approximation.

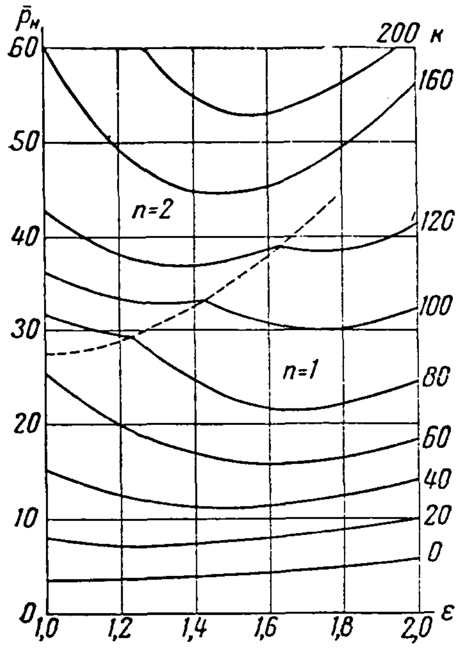

Fig. 5

For the isotropic shell, which is a particular case of the uniform orthotropic shell, the calculation was made and a graph of $\bar{p}_1 = f(\epsilon)$ was constructed. From the plot (Fig. 5), it may be seen that with an increase in the shell curvature parameter k, there is a change in the value of ϵ at which $\bar{p}_1$ is a minimum. Thus, for a plate (k = 0), the minimum occurs for $\epsilon = 1$, which corresponds to a square wave, while for k = 100, $\epsilon = 1.71$. At some curvature (in this case, for k = 113), it is found that there are two identical minimum values of $\bar{p}_1$ for $\epsilon_0 = 1.72$ and $\epsilon_0 = 1.37$, corresponding to the number of halfwaves along the arc n = 1 and n = 2. With an increase in the curvature (k > 113), the minimum value of $\bar{p}_1$ corresponds to n = 2, while ϵ_0 rises from 1.37 to 1.68 for k = 350 with an increase in curvature, where the equilibrium form with minimum $\bar{p}_1$ corresponds to n = 3.

We may conjecture that with a further increase in the curvature k, the value of ϵ_0 will tend to a definite magnitude close to 1.5.

From the data of Fig. 5, the curve $\bar{p}_1 = f_n(k)$ is constructed through the points corresponding to the minimum value of $\bar{p}_1$. The values of ϵ_0 corresponding to the minimum p_1 as a function of k are also shown.

Figure 6 shows that the coefficient χ_1 is equal to 0.26 for an isotropic shell of relatively large curvature; i.e.,

$$\text{for } k \geq 50 \quad \bar{p}_1 = 0.26k \text{ or } \sigma_1 = 0.26 \frac{Eh}{R}, \qquad \underline{/523} \qquad (120)$$

while for the shell of small curvature,

$$\text{for } k \leq 50 \quad \bar{p}_1 = 3.6 + 0.19\,k \text{ or } \sigma_1 = 3.6\,E\frac{h^2}{b^2} + 0.19\frac{Eh}{R} \qquad (121)$$

The plot of $\bar{p}_1 = f_n(k)$ (Fig. 6) is constructed for an isotropic shell but is entirely applicable to an orthotropic shell with the parameter $\psi = 1$, the only change in this case being the change in the quantity ϵ_0 corresponding to the minimum value of $\bar{p}_1$. This can easily be observed from the expressions for the coefficients of the system (55), (56). These coefficients depend on the quantity $\frac{\sqrt{4}}{\epsilon^2}$. With a change in the degree of orthotropicity φ, there is a change only in the value

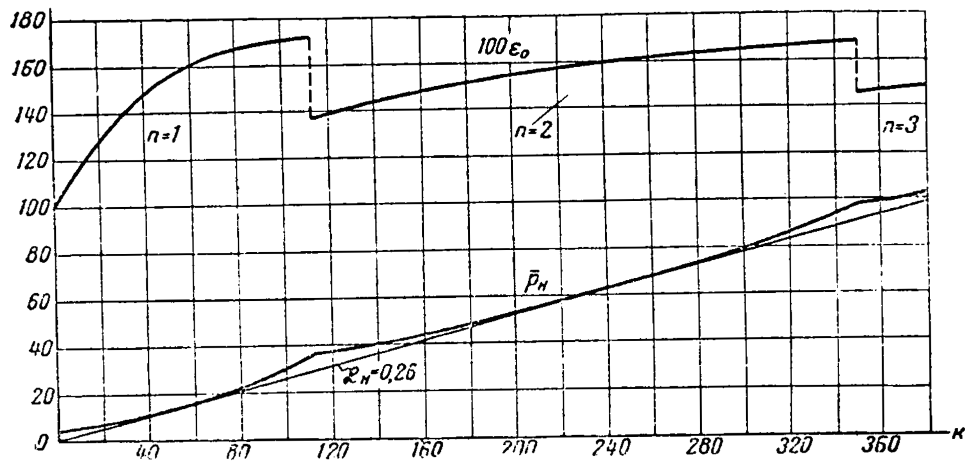

Fig. 6

of ϵ corresponding to $\overline{p}_1$, while the minimum value of $\overline{p}_1$ itself does not change. An m-fold increase of φ causes an $\sqrt[4]{m}$ fold increase of ϵ_0. Thus, for k = 60 for the isotropic shell, $\overline{p}_1 = 15.6$ for $\epsilon_0 = 1.6$ (Fig. 6), while for the uniform orthotropic shell ($\psi = 1$ for $\varphi = 16$), the lower critical stress parameter remains the same ($\overline{p}_1 = 15.6$), but the form of the buckling is characterized by more elongated buckles, namely, $\epsilon_0' = \epsilon_0 \sqrt[4]{\varphi} = 1.6 \sqrt[4]{16} = 3.2$.

For orthotropic shells with the parameter ψ different from unity, it is naturally not possible to use Fig. 6 or equations (120), (121). However, using the method described here, we can establish the relation of χ_1 as a function of ψ, which is suitable for the calculation of σ_1 for the shells of large curvature, and similar relations for shells of small curvature.

For circular cylindrical orthotropic shells which are non-uniform across the thickness, including the structurally orthotropic panels, it is possible by use of similar calculations to establish the relation for χ_1 as a function of the elastic parameters γ, ψ, and Ψ.

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